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AN ANALYSIS OF A SINGLE ITEM INVENTORY SYSTEM WITH RETURNS: THE--ETC(U)
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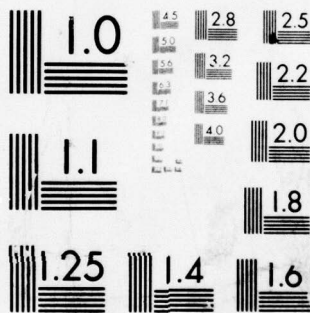
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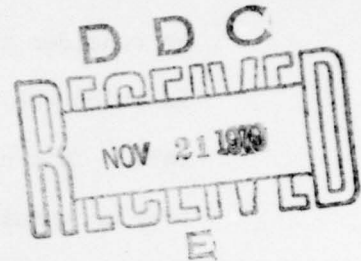
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July 1979



AN ANALYSIS OF A SINGLE ITEM INVENTORY SYSTEM
WITH RETURNS: THE SINGLE ECHELON CASE

by

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ABSTRACT

Inventory systems with returns are systems in which there are units returned in a repairable state, as well as demands for units in a serviceable state, where the return and demand processes are independent. We consider the inventory control of a single item at a single location in which the stationary return rate is less than the stationary demand rate. This necessitates an additional occasional procurement of units from an outside source. The objectives of this paper are to develop a cost model of this system managed under a continuous review procurement policy, and to develop a solution method for finding the policy parameter values. The key to the analysis is the use of a normally distributed random variable to approximate the steady-state distribution of net inventory.

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1. Introduction

Many models have been developed during the past 15 years pertaining to various aspects of managing repairable item inventory systems (e.g., [1],[4],[10],[11],[12],[15] and [16]).¹ Most of these models contain the assumption that the failure of a unit simultaneously generates a demand for a unit of exactly the same type, i.e. the demand process for serviceable units and the return processes of failed units are perfectly correlated.

In certain instances, however, this assumption of perfect correlation between the demand and return processes is not valid. For example, this can occur in situations where equipment is leased, rented, and/or sold, such as found in the telephone, computer and copying machine industries. Returns do not necessarily correspond to failures in these cases, but rather to lease or rental expirations. At the time a unit is returned, it may have to go through a repair or overhaul process before reissue. There is no reason to assume that with the expiration of a lease or rental agreement that the customer will request a unit of exactly the same type. Similarly, when a customer requests a particular type of unit, there is no reason to assume that the customer will return one of exactly the same type.

Although the assumption of partial dependence between the return and demand processes may, in general, be the most realistic, we will assume in our analysis that these processes are independent. This assumption was tested and found to be valid for the inventory system the

¹ A repairable item is an item which fails, but which can be repaired and subsequently made available to satisfy a future demand or an existing backorder.

authors examined. We will call inventory systems in which the return and demand processes for repairable items are independent, inventory systems with returns. It is this type of inventory system that will be analyzed in detail in this paper.

This section continues with a description of the inventory system we will study and concludes with a brief survey of the existing literature on this problem. In Section 2 we develop the stationary distribution of two key random variables that describe the probabilistic behavior of the inventory system. This analysis is used as the basis for a cost model presented in Section 3. In Section 4, we examine an important special case of our inventory system, and in Section 5, we conclude with a brief summary and some final comments.

The system we will study consists of a single type of item managed at a single location. A schematic representation of this inventory system is given by Figure 1. As shown, this location is assumed to contain both a repair facility for returned units, and a warehouse, or storage facility, for serviceable inventory.

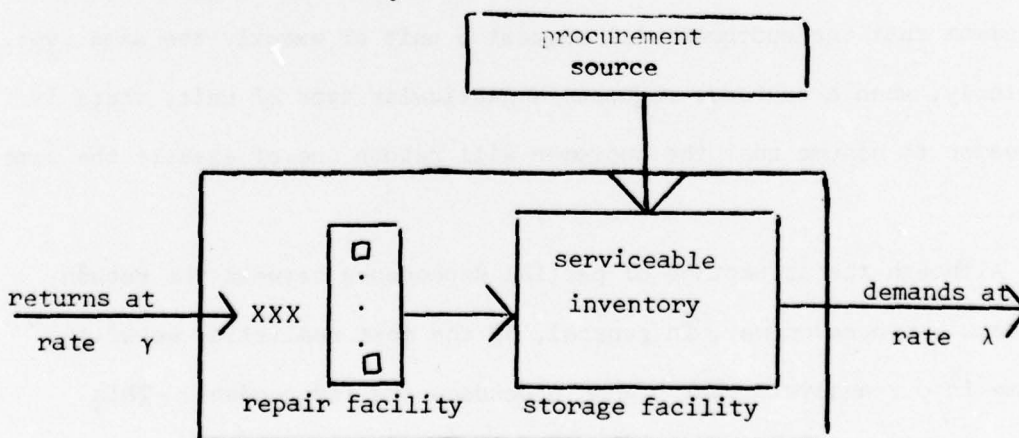


Figure 1. A Schematic Representation of the Inventory System

We assume returns of repairable units occur as a Poisson process with rate γ , and demands for serviceable units occur as a Poisson process with rate λ . These two processes are also assumed to be independent. γ is assumed to be less than λ , so that an occasional procurement of units from an outside source is required. Units procured in this manner arrive in a serviceable state τ time units after they are ordered.

The repair facility behaves as a first-come, first-served queueing system with Poisson arrivals (the Poisson returns). All returned units require repair, and repair times of returned units are independent. Since $\gamma < \lambda$, the repair system is always operating as long as repairables are present. No other assumptions about the queueing repair system (e.g. service time distribution, number of repair servers) are made.

The output of this queueing repair system is input to the stock of on-hand serviceable inventory, as is the arrival of outside procurement orders.

All demands that are not immediately satisfied are assumed to be back-ordered.

We define 'net inventory' at a point in time to be the number of on-hand serviceable units in the storage facility minus the number of outstanding backorders. We also define 'inventory position' at a point in time to be the sum of net inventory, the number of units in the repair queueing system, and the number of units on-order from the outside procurement source.

Let $I(t)$ = the inventory position at time t ,

$N(t)$ = the net inventory at time t ,

$R(t)$ = the number of units in the repair queueing system at time t ,

$P(t)$ = the number of units on-order from the outside supplier at time t ,

$\hat{O}(t)$ = the on-hand serviceable inventory at time t ,
 and $B(t)$ = the number of outstanding backorders at time t .

Then

$$I(t) = N(t) + R(t) + P(t),$$

and $N(t) = O(t) - B(t)$.

Our final assumption concerns the form of the procurement policy. We assume that a continuous review (Q, r) procurement policy is followed, i.e. when inventory position drops below $r + 1$, an order for Q units is immediately placed. Since the repair queueing system is assumed to be continuously operating, our objective is simply to find values of Q and r .

The objectives of this paper are to develop a model of this single item, single location inventory system with returns, and to present a procedure for finding the values of the policy parameters.

Only a few papers have been published on inventory systems with returns. These papers contain simplifying assumptions which make them of limited practical value. Heyman [6,7] considers optimal disposal policies for a single-item inventory system with returns; but his assumptions include instantaneous outside procurement (implying no backorders or lost sales) and no fixed cost of ordering (implying no lot-size reordering). Hoadley and Heyman [8] consider a two-echelon inventory system with outside procurement, returns, disposals, and transshipment; but their model is a one-period model, and all of the mentioned transactions are assumed to occur instantaneously. Simpson [16] develops the optimum solution structure for a finite-horizon, periodic review model of an inventory system with

returns. His model allows for correlation between the return and demand processes. Backlogging is permitted, but both repairs and outside procurements are assumed to be instantaneous.

For the most part, the methods of analysis in these three papers rely heavily upon the assumptions of instantaneous repair and procurement. Their approaches are of little use when analyzing situations in which repair and procurement times are not zero.

Finally, Schradly [14] solves for repair carcass and procurement lot-sizes for a completely deterministic system. Gajdalo [2] extends this to a 'continuous review repair policy' for an inventory system with stochastic (compound Poisson) returns and demands. He uses computer simulation to test several heuristics for computing the reorder point and reorder lot-sizes for both procuring and repairing items. All lead-times, including repair times, are assumed constant.

2. Analysis

The analysis begins with the derivation of the steady-state distribution of inventory position. This result is used in the derivation of an approximation of the steady-state distribution of net inventory, and is followed by a discussion of the accuracy of the approximation.

2.1 Derivation of the Stationary Distribution of Inventory Position

Changes in the state of the inventory position are caused only by demands and returns. State i ($i = r+1, r+2, \dots$) can be entered from state $i + 1$ when a demand for a serviceable item occurs; state j ($j = r+2, r+3, \dots$) can be entered from state $j - 1$ when an item is

returned. In addition, state $r + Q$ can also be reached from state $r + 1$ when a serviceable item is demanded (an order for Q units is placed immediately when the inventory position drops below $r + 1$). The time between state transitions is exponentially distributed, since the return and demand processes are Poisson processes. The state transition flow diagram is given in Figure 2, with the transition rates as indicated.

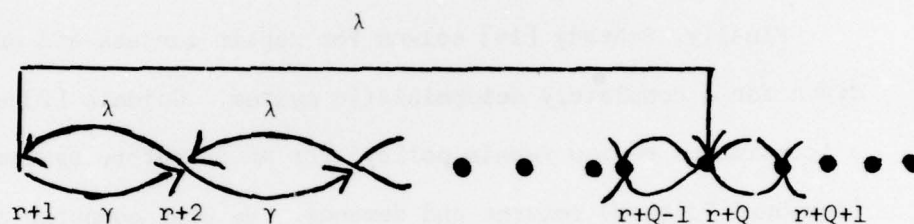


Figure 2. State Transition Flow Diagram For Inventory Position

Let $u_i = \lim_{t \rightarrow \infty} \text{Prob}(I(t) = r + 1 + i)$, the stationary probability that inventory position is equal to $r + 1 + i$. This limit exists because the states of this system are the states of an irreducible, ergodic, Markov chain [13]. The steady-state balance equations corresponding to this system are

$$\begin{aligned}
 (\lambda + \gamma)u_0 &= \gamma u_1, \\
 (\lambda + \gamma)u_1 &= \gamma u_0 + \lambda u_2, \\
 (\lambda + \gamma)u_2 &= \gamma u_1 + \lambda u_3, \\
 &\vdots \\
 (\lambda + \gamma)u_{Q-1} &= \gamma u_{Q-2} + \lambda u_Q + \lambda u_0, \\
 (\lambda + \gamma)u_Q &= \gamma u_{Q-1} + \lambda u_{Q+1}, \\
 &\vdots
 \end{aligned}
 \tag{1}$$

A generating function approach can be used to solve for the u_i . Define the generating function $G(z)$ to be $G(z) = \sum_{i=0}^{\infty} z^i u_i$. Using (1) we find that

$$(2) \quad G(z) = \frac{\lambda - \gamma}{Q} \frac{(1 - z^Q)}{(1 - z)(\lambda - \gamma z)},$$

from which we find that the u_i are given by

$$(3) \quad u_i = \begin{cases} \frac{1 - (\frac{\gamma}{\lambda})^i}{Q}, & i \leq Q - 1 \\ \frac{(\frac{\gamma}{\lambda})^{i-Q+1} [1 - (\frac{\gamma}{\lambda})^Q]}{Q}, & i \geq Q \\ 0, & \text{otherwise,} \end{cases}$$

and the mean and variance of the stationary distribution of inventory position are given by

$$(4) \quad E[\lim_{t \rightarrow \infty} I(t)] = r + 1 + G'(1) = r + 1 + \frac{Q-1}{2} + \frac{\gamma}{\lambda - \gamma},$$

and

$$(5) \quad \text{Var}[\lim_{t \rightarrow \infty} I(t)] = G''(1) + G'(1) - [G'(1)]^2 = \frac{Q^2 - 1}{12} + \frac{\lambda \gamma}{(\lambda - \gamma)^2},$$

respectively.

If $Q = 1$, Figure 2 is the transition flow diagram for an M/M/1 queueing system in which the 'arrival' rate is γ , and the 'service' rate is λ . In this case (3) reduces to the geometric distribution, which is the well-known steady state distribution of the number of customers present in an M/M/1 system.

Note that when $\gamma = 0$, (4), (5), and (3) reduce to the mean variance, and probability distribution, respectively, of a uniformly distributed random variable, a well known result (see reference 5).

2.2 An Approximation to the Stationary Distribution for Net Inventory

Next, we develop an approximation to the stationary distribution of net inventory, which is the basis for the cost model used to determine optimal values of Q and r .

Recall that τ , the procurement lead-time, is constant. Thus, any units on-order at time $t - \tau$ will have arrived by time t . Similarly, any order placed after time $t - \tau$ will not have arrived by time t . Therefore, we see that

$$(6) \quad N(t) = I(t-\tau) - R(t-\tau) + Z(t-\tau, t) - D(t-\tau, t),$$

where $R(t-\tau)$ = the number of units in the repair system at time $t - \tau$,

$Z(t-\tau, t)$ = the output of the repair system in the interval $(t-\tau, t]$,

and $D(t-\tau, t)$ = the number of demands in the interval $(t-\tau, t]$.

$R(t-\tau)$ is subtracted from $I(t-\tau)$ so that we do not double count the units in the repair system at time $t - \tau$ that complete service by time t . Therefore, net inventory at time t consists of units on-order or already serviceable at time $t - \tau$ (both measured in $I(t-\tau)$), plus those units completing repair by time $t - \tau$, minus demands over the interval $(t-\tau, t]$.

Let us separately examine the individual terms of (6). The steady-state distribution of $I(t-\tau)$ has already been obtained. The number of demands over the interval $(t-\tau, t]$ is Poisson distributed with mean $\gamma\tau$ and is independent of the other three random variables on the right-hand side of equation (6).

The distributions of $R(t-\tau)$ and $Z(t-\tau, t)$ are readily available for many queueing systems; but, they are not independent of each other or of $I(t-\tau)$. The number in the repair system at time $t - \tau$, $R(t-\tau)$, clearly depends on the inventory position at time $t - \tau$, $I(t-\tau)$. The output of the queueing system in $(t-\tau, t]$, $Z(t-\tau, t)$, depends on the number in the repair system at the start of the interval, $R(t-\tau)$. This latter dependence decreases as τ increases, and one could assume as an approximation that $R(t-\tau)$ and $Z(t-\tau, t)$ are independent for values of τ that are large relative to the mean repair time. However, the dependence between $R(t-\tau)$ and $I(t-\tau)$ cannot be ignored. The joint distribution of $R(t-\tau)$ and $I(t-\tau)$ is difficult to develop analytically. Consequently, an approximation to the distribution of net inventory will be developed, using (6), rather than developing the exact distributions.

The normal distribution was chosen to be a continuous approximation to the steady-state distribution of net inventory. This was done for two reasons. First, we observed that the steady state distribution of net inventory for several test cases (obtained via simulation) resembled a normal distribution. Second, a theoretical basis exists for the use of the normal distribution. Note that when the effect of the return and repair processes are small, $I(t-\tau)$ is close to being uniformly distributed. Also,

the Poisson random variable $D(t-\tau, t)$ is well approximated by a normally distributed random variable if $\lambda\tau > 10$. Then $N(t)$ is close to being the difference between a uniform random variable and a normally distributed random variable, which itself is close to being a normally distributed random variable.

Equation (6) is used to determine the mean μ and to approximate the variance, σ^2 , of this normal distribution. Letting $t \rightarrow \infty$, we have

$$\begin{aligned}
 \mu &= E(N(t)) = E(I(t-\tau)) - E(R(t-\tau)) \\
 (7) \quad &+ E(Z(t-\tau, t)) - E(D(t-\tau, t)) \\
 &= r + 1 + \frac{Q-1}{2} + \frac{\gamma}{\lambda-\gamma} - E(R(t-\tau)) + \gamma\tau - \lambda\tau,
 \end{aligned}$$

using (5), and noting that the expected output of a queueing system over an interval is equal to the expected input over an interval of the same length. Also, by ignoring covariance terms we approximate σ^2 by

$$\begin{aligned}
 \sigma^2 &= \text{Var}(N(t)) \approx \text{Var}(I(t-\tau)) + \text{Var}(R(t-\tau)) \\
 (8) \quad &+ \text{Var}(Z(t-\tau, t)) + \text{Var}(D(t-\tau, t)) \\
 &= \frac{Q^2-1}{12} + \frac{\lambda\gamma}{(\lambda-\gamma)^2} + \text{Var}(R(t-\tau)) + \text{Var}(Z(t-\tau, t)) + \lambda\tau,
 \end{aligned}$$

using (5). Note that exact expressions and good approximations for $E(R(t-\tau))$, $\text{Var}(R(t-\tau))$, and $\text{Var}(Z(t-\tau, t))$ are available for many queueing systems (e.g. see [3]).

The accuracy of the normal approximation, whose mean and variance are given by (7) and (8), was tested using an incomplete factorial experiment. The variable factors were the number of repair servers, the

repair service distribution, the repair system traffic intensity, the procurement lead-time, τ , the procurement lot-size Q , and the ratio γ/λ . In each test case, the accuracy of the normal approximation, was measured by finding the area between the normal curve and the curve representing the continuous version of the distribution of net inventory which was obtained via simulation.

The conclusion drawn from this experiment was that the major factor affecting the accuracy of the normal approximation is the ratio of the return rate to the demand rate, γ/λ . In fact, the normal approximation is quite accurate when $\gamma/\lambda < .6$. However, a visual inspection of the normal curves revealed that the normal approximation was a good one in the left-hand tail of the distribution of net inventory in all the test cases. As in most inventory problems, the tail of the distribution is all that is needed to determine optimal values for Q and r . (The reason for this will be discussed in Section 3.) Thus the normal approximation was deemed to be an acceptable approximation for all realistic situations.

3. Cost Model and Solution Method

The optimization model we wish to study includes a fixed procurement order cost, a holding cost, and a time-weighted backorder cost. In particular, let

A = the fixed procurement order cost (\$/procurement order),

h = the holding cost (\$/unit-year),

and $\hat{\pi}$ = the backorder cost (\$/unit-year).

Our objective function K , will be the expected sum of annual procurement ordering costs, holding costs, and backorder costs. It will be evaluated by taking the sum of

- 1) $A \times$ (the expected number of orders placed per year),
- 2) $h \times$ (the expected serviceable on-hand inventory at a random point in time),
- and 3) $\hat{\pi} \times$ (the expected number of outstanding backorders at a random point in time).

Both the expected on-hand inventory and expected backorders at a random point in time will be calculated using the normal approximation to the distribution of net inventory.

Note that we need not consider holding costs charged against units in repair. Due to the assumption that no inserted idleness in the queueing repair system is allowed, these holding costs are independent of the values of the procurement policy parameters.

Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the standard normal density and standard normal cumulative functions, respectively; i.e. let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}, \text{ and}$$

$$\Phi(x) = \int_{t=-\infty}^x \phi(t)dt.$$

Let $h(x)$ be the normal density, which is the continuous approximation to the steady-state distribution of net inventory, and which has mean μ and variance σ^2 given by (7) and (8), respectively. Thus the expected number of backorders at any point in time is

$$(9) \quad \sigma\phi\left(\frac{\mu}{\sigma}\right) - \mu\Phi\left(-\frac{\mu}{\sigma}\right),$$

as is derived in reference 5.

Since

$$\begin{aligned} E(\text{on-hand inventory}) &= E(\text{inventory position}) + E(\text{backorders}) \\ &\quad - E(\text{number in repair}) - E(\text{number on-order}), \end{aligned}$$

we have the expected on-hand inventory equal to

$$(10) \quad r + 1 + \frac{Q-1}{2} + \frac{\gamma}{\lambda-\gamma} + \sigma\phi\left(\frac{\mu}{\sigma}\right) - \mu\Phi\left(-\frac{\mu}{\sigma}\right) - E(R(t)) - (\lambda-\gamma)\tau.$$

Note that the last term, the expected amount on-order at any point in time, is equal to the rate at which demands are ultimately met by outside procurement, $\lambda - \gamma$, times the constant procurement lead-time, τ .

In what follows, it will be easier to think of μ and σ^2 as functions of r and Q . Specifically, let

$$(11) \quad \mu = r + \frac{Q}{2} + c$$

and

$$(12) \quad \sigma^2 = \frac{Q^2}{12} + d,$$

where

$$(13) \quad c = \frac{\gamma}{\lambda-\gamma} + \frac{1}{2} - E(R(t)) - (\lambda-\gamma)\tau$$

and

$$(14) \quad d = \frac{\lambda\gamma}{(\lambda-\gamma)^2} - \frac{1}{12} + \text{Var}(R(t)) + (\lambda+\gamma)\tau.$$

Finally, the rate at which demands are met by outside procurement, $\lambda - \gamma$, divided by Q , the procurement lot-size, gives the expected number of procurement orders placed per year.

Combining our previous results, we see that the objective function can be expressed as

$$\begin{aligned}
 K &= \frac{(\lambda - \gamma)A}{Q} + \pi \left[\sigma \Phi\left(\frac{\mu}{\sigma}\right) - \mu \Phi\left(-\frac{\mu}{\sigma}\right) \right] \\
 (15) \quad &+ h \left[r + \frac{1}{2}Q + \sigma \Phi\left(\frac{\mu}{\sigma}\right) - \mu \Phi\left(-\frac{\mu}{\sigma}\right) + c \right] \\
 &= \frac{(\lambda - \gamma)A}{Q} + (\pi + h) \left[\sigma \Phi\left(\frac{\mu}{\sigma}\right) - \mu \Phi\left(-\frac{\mu}{\sigma}\right) \right] + h \left(r + \frac{1}{2}Q + c \right),
 \end{aligned}$$

where c is given by (13).

This objective function is not convex in Q , but is convex in r . This is easily proven by showing that the backorder function $\sigma \Phi\left(\frac{\mu}{\sigma}\right) - \mu \Phi\left(-\frac{\mu}{\sigma}\right)$ is convex in μ . Since $\mu = r + \frac{Q}{2} + c$, a function that is convex in μ is also convex in r . Setting $\frac{\partial K}{\partial r} = 0$, we see that

$$(16) \quad \Phi\left(-\frac{\mu}{\sigma}\right) = \frac{h}{\pi + h}.$$

Recall, from equations (11) and (12), that the mean μ of our approximating normal density to net inventory is a function of r , while the variance, σ^2 , is not. Thus, for a fixed value of Q , the variance of the normal distribution representing net inventory is fixed. Only the mean, or 'location' of the curve, is decided by choosing a value of r . Therefore, equation (16) indicates that once the variance is fixed, the 'location' of the normal curve should be chosen so that the cumulative area to the left of the y-axis is $\frac{h}{\pi + h}$, as illustrated in Figure 3.

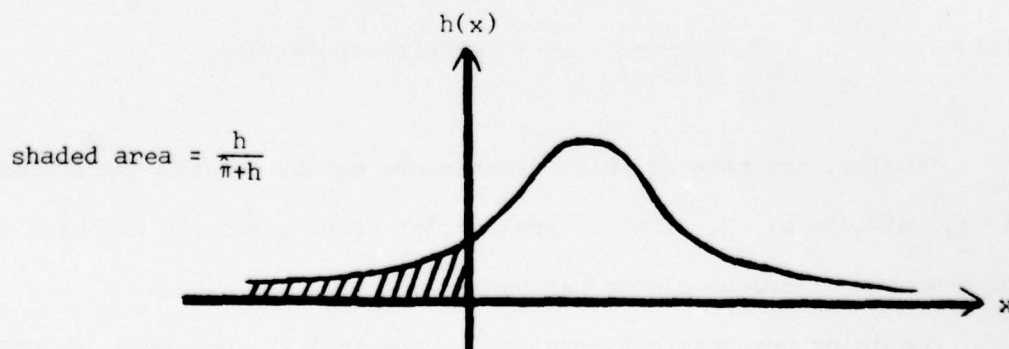


Figure 3. Location of the Normal Curve.

In most real situations, the backorder cost $\hat{\pi}$ is large compared to the holding cost h . This makes the fraction $\frac{h}{\hat{\pi}+h}$ small. Recall that this fraction is the area to the left of the y -axis under the normal curve. The expected number of backorders is calculated using equation (9), and the expected on-hand inventory is calculated in equation (10) also using (9). Thus, accuracy of the normal approximation is required only in the tail of the distribution, since $\frac{h}{\hat{\pi}+h}$ is usually small.

Returning to (16) and rewriting it in terms of r^* and Q^* , the optimal values of Q and r , respectively, we have

$$\Phi\left(\frac{r^* + \frac{Q^*}{2} + c}{\sqrt{\frac{(Q^*)^2}{12} + d}}\right) = \frac{\hat{\pi}}{\hat{\pi}+h}$$

or

$$(17) \quad r^* = \sqrt{\frac{(Q^*)^2}{12} + d} \Phi^{-1}\left(\frac{\hat{\pi}}{\hat{\pi}+h}\right) - \frac{1}{2} Q^* - c.$$

For a fixed value of Q , the optimal value of r is given by equation (17).

To find the optimal value of Q , one can rewrite equation (15) in terms of r and Q . Using equation (17) to write the objective function solely as a function of Q , (15) simplifies to

$$(18) \quad K = \frac{(\lambda - \gamma)A}{Q} + (\hat{\pi}+h)\sqrt{\frac{Q^2}{12} + d} \cdot \Phi\left(\Phi^{-1}\left(\frac{\hat{\pi}}{\hat{\pi}+h}\right)\right).$$

This can be seen to be a convex function of Q . While the original objective function, K , is not convex everywhere in both Q and r , upon deriving an optimality condition (17), K is convex in both Q and r over the region of interest. Setting $\frac{dK}{dQ} = 0$, we find that Q^* is the value of Q that satisfies

$$(19) \quad \frac{Q^3}{\sqrt{\frac{Q^2}{12} + d}} = \frac{12(\lambda - \gamma)A}{\alpha},$$

where

$$(20) \quad \alpha = (\hat{\pi} + h) \phi \left(\phi^{-1} \left(\frac{\hat{\pi}}{\hat{\pi} + h} \right) \right).$$

A Fibonacci search may be used to find Q^* since the function on the left side of (19) increases with Q . (Note, also, the similarity to the usual lot-size formula. Ignoring some of the constants, (19) is roughly of the form

$$Q = \sqrt{\frac{(\lambda - \gamma)A}{h}} \cdot \text{constant} .)$$

This relation is independent of r . Thus, once Q^* is found, r^* is found using (17).

We conclude this section with a numerical example. Suppose that demands for an item occur as a Poisson process with rate $\lambda = 600/\text{year}$. The cost of holding a serviceable unit in on-hand inventory is $h = \$200/\text{year}$; the back-order cost is $\hat{\pi} = \$800/\text{backordered unit per year}$; the cost of placing an order is $A = \$1000/\text{order}$; the procurement lead-time is $\tau = .1$ years; returns occur as a Poisson process, with rate $\gamma = 500/\text{year}$, to a single-server repair

system with an exponential service time distribution with rate $\eta = 600$ units per year.

The solution procedure starts by calculating the expected number in repair and the variance of the number in repair. For an M/M/1 system, the expected number in repair is $\frac{\gamma}{\eta - \gamma} = 5$, and the variance of the number in repair is $\frac{\gamma/\eta}{(1 - \frac{\gamma}{\eta})^2} = 30$.

Using (13) and (14), $c = -9.5$, $d = 169.9$, and $\alpha = 280.3$. Then, Q^* satisfies

$$\frac{Q^3}{\sqrt{\frac{Q^2}{12} + 169.9}} = \frac{12(100)(1000)}{280.31} = 4281.0, \text{ or}$$

to the nearest tenth, $Q^* = 42.6$. From this and (17),

$$r^* = \sqrt{\frac{(42.6)^2}{12} + 169.9} \Phi^{-1}(.8) - \frac{42.6}{2} + 9.5 = 3.3.$$

If Q^* and r^* are required to be integer, the total cost K can be evaluated for the rounded combinations of these values. In this case, the optimal combination is $Q^* = 43$ and $r^* = 3$. For these values, $K = \$7356$. Also, $\mu = 15.0$ and $\sigma = 18.0$, so that the expected backorders at any point in time, $\sigma\phi(\frac{\mu}{\sigma}) - \mu\Phi(-\frac{\mu}{\sigma})$, is equal to 2.03. The total cost of \$7356 consists of \$2326 for ordering, \$1624 for backorders, and \$3406 for holding serviceable stock.

4. The Special Case $\gamma = 0$

A special case of the system described in Section 1 is one in which $\gamma = 0$. Without returns of repairable items, the model simplifies to the standard (Q, r) model with Poisson demands and a constant procurement lead-time. A common method of solution to this problem is found in Chapter 4 of reference 5. This method, which will be referred to as the 'standard' method (SM), will be compared to the solution method derived previously for the case $\gamma = 0$. The latter will be referred to as the normal approximation method (NAM).

The SM is based on the assumption that the probability that demand over a lead-time exceeds $r + Q$ is negligible. This approximation results in the following simplified objective function which is jointly convex in Q and r :

$$(21) \quad K = \frac{\lambda A}{Q} + h\left(\frac{Q}{2} + \frac{1}{2} + r - \lambda \tau\right) + \frac{1}{Q}(\hat{\pi} + h)\beta(r),$$

where

$$(22) \quad \beta(v) = \sum_{u=v+1}^{\infty} (u-v-1)\underline{P}(u; \lambda \tau)$$

$$= \frac{(\lambda \tau)^2}{2} \underline{P}(v-1; \lambda \tau) - \lambda \tau v \underline{P}(v; \lambda \tau) + \frac{v(v+1)}{2} \underline{P}(v+1; \lambda \tau),$$

$$(23) \quad \underline{P}(a, \mu) = \sum_{x=a}^{\infty} p(x; \mu),$$

and

$$(24) \quad p(x; \mu) = e^{-\mu} \mu^x / x!$$

The costs h , $\hat{\pi}$, and A are defined as before.

The optimal value for r , call it r^* , is the largest integer satisfying

$$(25) \quad rp(r; \lambda\tau) - (r - \lambda\tau)P(r; \lambda\tau) > \frac{hQ}{\pi + h},$$

and the optimal value for Q , call it Q^* , is the largest integer satisfying

$$(26) \quad Q(Q-1) < \frac{2\lambda}{h} \left[A + \frac{\hat{\pi} + h}{\lambda} \beta(r) \right], \text{ or } Q = 1.$$

The SM is an iterative algorithm. The algorithm normally starts with $Q = \sqrt{\frac{2\lambda A}{h}}$. A value for r is found using (25), and a new value for Q is found using (26). This procedure continues until the values of Q and r do not change.

The NAM may be applied simply by setting $\gamma = 0$. Since inventory position is uniformly distributed between $r + 1$ and $r + Q$ when $\gamma = 0$, the mean and variance of the stationary distribution for inventory position are $r + \frac{Q}{2} + \frac{1}{2}$ and $\frac{Q^2 - 1}{12}$, respectively. The net inventory at time t , $N(t)$, is given by

$$(27) \quad N(t) = I(t - \tau) - D(t - \tau, t).$$

Then the mean and variance of net inventory can be expressed as

$$\mu = r + \frac{Q}{2} + c \text{ and } \sigma^2 = \frac{Q^2}{12} + d, \text{ where}$$

$$(28) \quad c = \frac{1}{2} - \lambda\tau$$

and

$$(29) \quad d = \lambda\tau - \frac{1}{12}.$$

Furthermore, the variance in this case is exact. With only these small changes in the constants c and d , r^* and Q^* are found as before using (17) and (19).

The biggest advantage of using this normal approximation method with $\gamma = 0$ is that an iterative procedure is not required to find Q^* and r^* . A binary or Fibonacci search quickly determines the value of Q^* , from which r^* is uniquely determined.

A factorial experiment was run to test the NAM versus the SM, and to determine which factors influence this comparison. Five different values for each of the factors $\hat{\pi}$, λA , and $\lambda\tau$ were used, resulting in $5^3 = 125$ test cases. The values used are listed in Table 1.

Table 1. Data for Factorial Experiment

$\hat{\pi}$	λA	$\lambda\tau$
100	1,000	5
200	2,000	10
500	5,000	25
1,000	10,000	50
2,000	20,000	100

The holding cost was fixed at $h = 100$.

Recall that the SM is based on an approximation to the objective function, as is the NAM. Thus, in order to test the solutions generated by the two methods, we use an objective function which is the exact measure of the expected ordering, holding, and backorder costs. This function is

$$(30) \quad K = \frac{\lambda A}{Q} + h\left(\frac{Q}{2} + \frac{1}{2} + r - \lambda\tau\right) + \frac{1}{Q}(\hat{\pi}+h)[\beta(r) - \beta(r+Q)],$$

where $\beta(v)$ is given by (22). (This expression is derived by Hadley and Whitin [5].) Note that the term $\frac{1}{Q}(\hat{\pi}+h)[\beta(r+Q)]$ was dropped from the objective function in the SM in order to simplify the analysis.

The results of the experiment may be summarized as follows. The NAM yielded an expected annual total cost (using K as defined in (30)) less than or equal to that obtained by the SM in 110 of the 125 cases, or 88% of the time. It yielded a value for K which was strictly less than that obtained by the SM in 95 cases, or 77% of the time. In these 95 cases, the average reduction in total cost was approximately 1% with the extreme case having a reduction of 10%. In the 15 cases in which the normal approximation method yielded a higher value of K , the average increase was 2.5%, with the extreme case having an increase of 6%.

The only factor which had a noticeable effect on the difference of the results between the two methods was $\lambda\tau$, the lead-time demand. This is explained by the fact that the normal distribution becomes a better continuous approximation to the Poisson distribution as the mean of the Poisson distribution increases. Thus, as $\lambda\tau$ increases, the normal distribution becomes a better continuous approximation to the stationary distribution of net inventory.

To illustrate these observations, in the 25 cases in which $\lambda\tau = 5$, the normal approximation method yields a higher value of K in eight of the cases, or 32% of the time. The average increase in K in these eight cases is 2%, and the largest increase is 6%. In the 25 cases in which $\lambda\tau = 10$, the normal approximation method yields a higher value of K in only five of the cases, or 20% of the time. Here, in these five cases, the average increase is only 1% with the largest increase being 3%. Yet of the remaining 75 cases, in which we have $\lambda\tau \geq 25$, the normal approximation method yielded a higher value of K in only 2 of these cases.

Recall, also, that the SM is an iterative one. The average number of iterations necessary to solve these 125 test cases using the SM was 3.14. The number of iterations ranged from two to ten, and increased with an increase in $\lambda\tau$. In the 50 cases in which $\lambda\tau \geq 50$, the average number of iterations was 4.18, and rose to 4.90 in the 25 cases in which $\lambda\tau = 100$.

The difference in computation time on an IBM 370/168 also increased as $\lambda\tau$ increased. The 25 test cases in which $\lambda\tau = 5$ took .24 seconds of execution time to solve via the NAM, and took .29 seconds with the SM. In the 25 test cases in which $\lambda\tau = 50$, however, the NAM took .23 seconds while the SM required .62 seconds.

In conclusion, the use of the NAM should be determined by the magnitude of $\lambda\tau$. In the 25 cases in which $\lambda\tau = 5$, the two methods yielded virtually identical values of K (20 of the 25 cases had values of K within 1% of each other). Also, only two iterations were needed using the SM in 24 of the 25 cases. Thus, for $\lambda\tau$ small, the two methods appear to be about equal in accuracy and computational complexity.

However, as soon as $\lambda\tau$ exceeds 10, both the accuracy of the NAM improves, and the number of iterations necessary to solve for Q^* and r^* using the standard method increases. Thus, the NAM is recommended for this cost model when $\lambda\tau \geq 10$ for both accuracy and computational reasons.

5. Summary and Concluding Comments

We have shown how to find near-optimal parameter values for a procurement policy for certain inventory systems with returns. The key was the use of a normal approximation to the steady-state distribution of net inventory. This led to the solution of a cost model which was convex in the procurement parameters. When $\gamma = 0$, this solution method provides an accurate and efficient alternative to the currently accepted iterative algorithm for the standard Poisson demand, constant lead-time inventory problem without returns.

In our model, we assumed the procurement policy to be a stationary (Q, r) policy. This policy is by no means the optimal one. In reference 9, it is shown that, for the special cases of $M/M/1$ and $M/G/\infty$ queueing repair systems, one can lower total expected costs by redefining inventory position, and allowing variable reorder points as follows. Inventory position is redefined to be net inventory plus the number of units on-order. The analysis proceeds exactly as described in Section 2 (with some of the constants redefined). This results in a reduction in σ^2 , the variance of net inventory, since the variance of the number of units in repair is no longer included in σ^2 . The reorder point, expressed in terms of

inventory position, is then a function of the number of units in repair, rather than a constant. Reductions in total expected costs can be achieved by using a state dependent reorder point when the variance of the number of units in repair is very large. A 10% reduction in total expected cost was achieved using the variable reorder point policy in an M/M/1 repair system with traffic intensity $\rho = 499/500$. This is an extreme case, however. The average annual cost of using the stationary (Q,r) policy was within 1% of the average annual cost obtained using the non-stationary one in almost all test cases. Since this is the case, and since a stationary (Q,r) policy is easy to use, the stationary (Q,r) policy is an attractive policy to implement.

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20. Abstract - continued

system managed under a continuous review procurement policy, and to develop a solution method for finding the policy parameter values. The key to the analysis is the use of a normally distributed random variable to approximate the steady-state distribution of net inventory.

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